Introduction to Dynamic Programming

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November 10, 2019

Swiss Olympiad in Informatics

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Introduction

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- a technique for solving problems (in particular optimization problems)

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Dynamic programming is...

- **not** an algorithm
- a technique for solving problems (in particular optimization problems) more efficiently.

What is dynamic programming: A guideline

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What is dynamic programming: A guideline

Keep it simple, stupid!

- Doing the same thing twice is bad.
- When you use the same code more than once, you use a function.
- Your program should also be lazy and avoid to compute what it has already computed.

What is dynamic programming

DP is a technique that you may use when you can divide a problem into **subproblems** and build the full solution using the partial solutions (using **recursion**), but the subproblems overlap and you end up solving the same subproblems over and over again.

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DP is a technique that you may use when you can divide a problem into **subproblems** and build the full solution using the partial solutions (using **recursion**), but the subproblems overlap and you end up solving the same subproblems over and over again.
A dynamic program avoids this problem by **remembering** what it has already done and not computing it again.

A simple example: Fibonacci's sequence

Fibonacci's sequence

Fibonacci's sequence can be defined as follows:

$$\begin{cases} f_0 = 0 \\ f_1 = 1 \\ \forall n \in \mathbb{N} \setminus \{0, 1\}, f_n = f_{n-1} + f_{n-2} \end{cases}$$

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Here are a few first values of the sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

Fibonacci's sequence: Intuitive algorithm

An intuitive way of computing values of the sequence would be:

```
int fib(int n) {
    if(n==0) return 0;
    if(n==1) return 1;
    return fib(n-1) + fib(n-2);
}
```

Fibonacci's sequence: The problem

If you try to run this code to compute f_{100}, you'd have to be **very** patient to get an answer.

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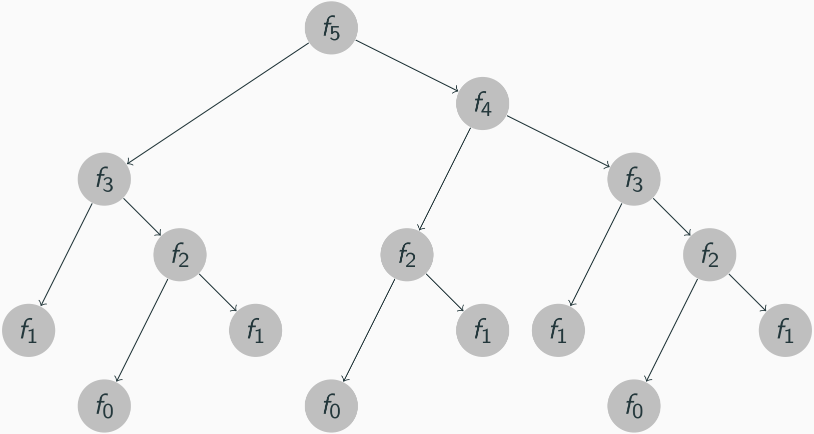
Why?

Fibonacci's sequence: The problem

Our program computes the same values over and over.

Look at this tree:



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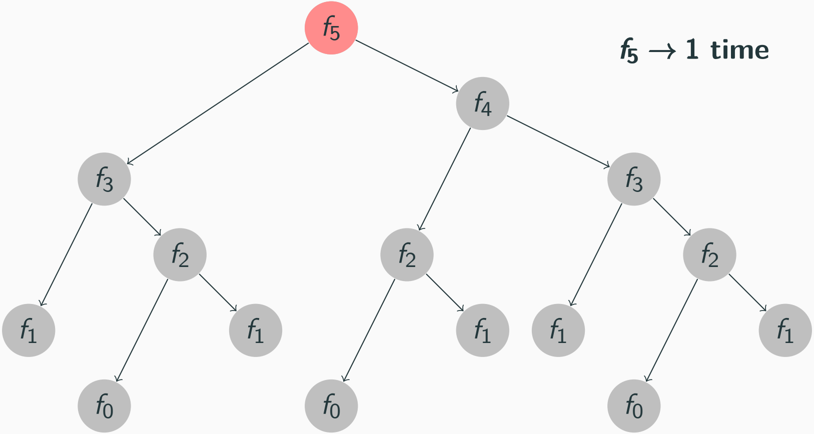

$f_5 \rightarrow$ 1 time

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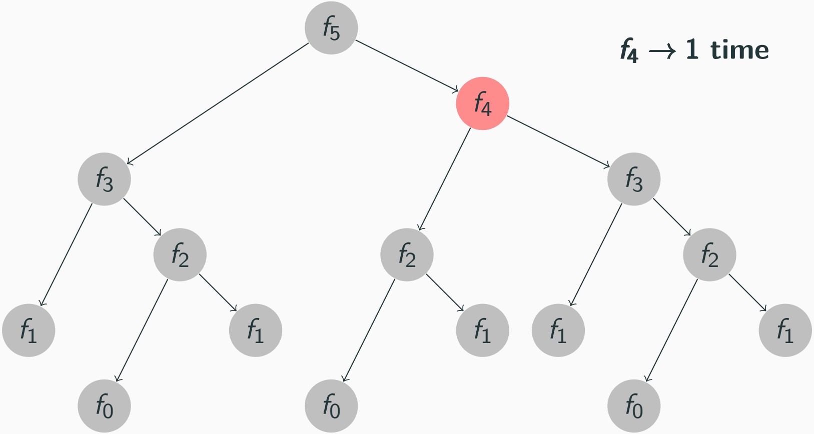

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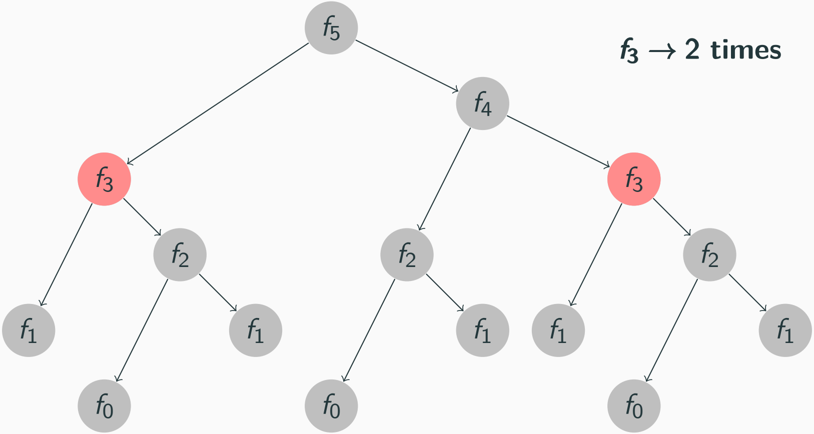

f_3 → 2 times

Fibonacci's sequence: The problem

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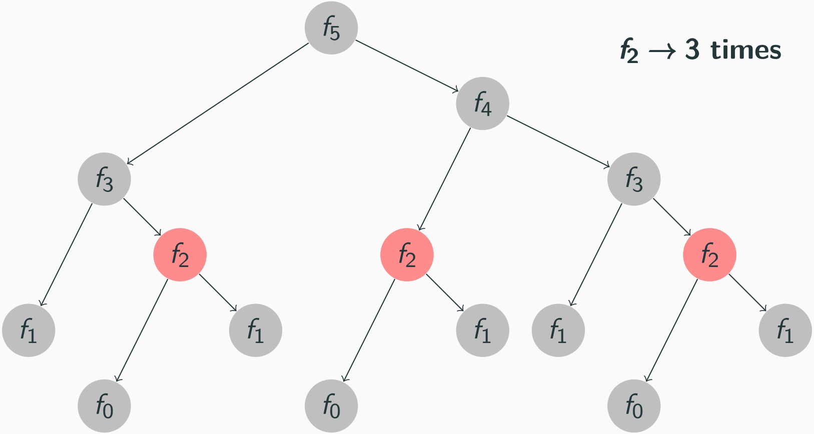

f_2 → 3 times

Fibonacci's sequence: The problem

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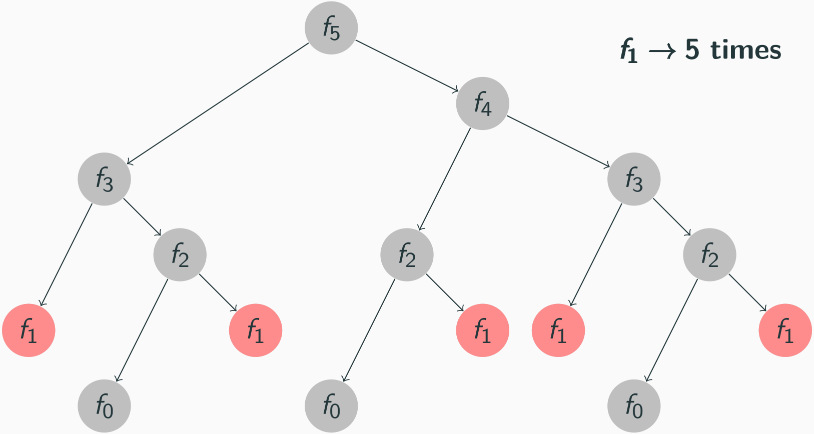

f_1 → 5 times

Fibonacci's sequence: The problem

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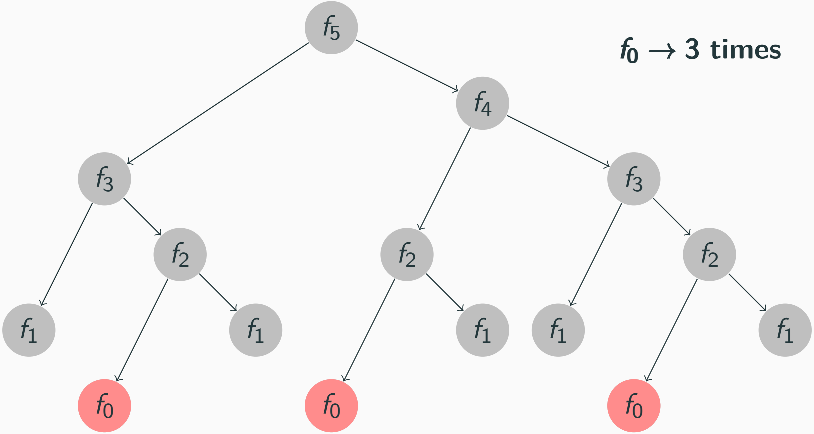

f_0 → 3 times

Fibonacci's sequence: The problem

This example shows us that the number of operations increases at the same speed as the results (the number of times we compute each value are all increasing values of Fibonacci's sequence)!

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The Fibonacci sequence grows exponentially, so we have an exponential running time...

Fibonacci's sequence: The solution

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We can do (much) better. We don't need to compute anything twice.

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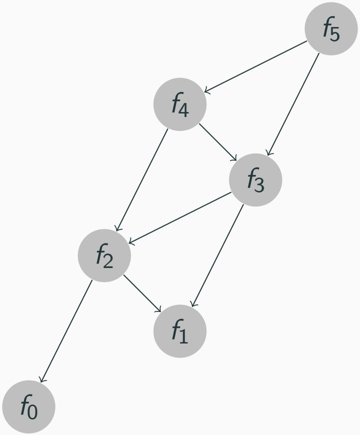
We can do (much) better. We don't need to compute anything twice.

We choose to trade memory usage for speed: store what we have computed and check if we have already computed the answer for each call.

Fibonacci's sequence: The solution

We always check whether we already have the solution for a subproblem:



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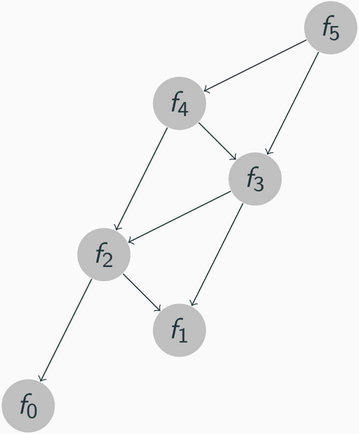
We always check whether we already have the solution for a subproblem:

$f_5 = f_4 + f_3$



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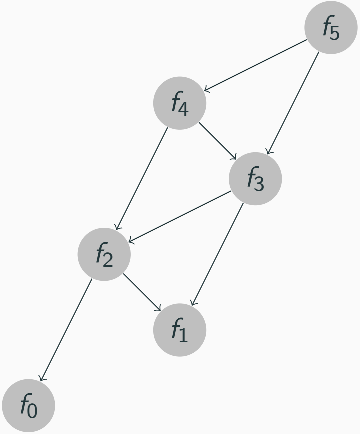

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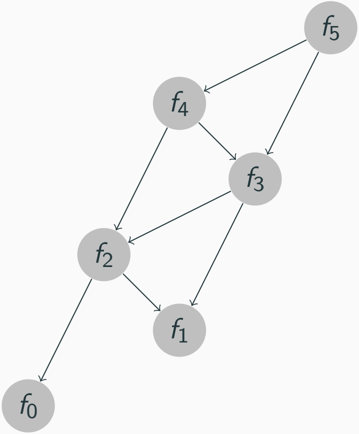

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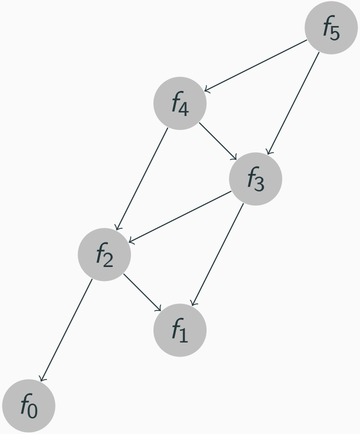

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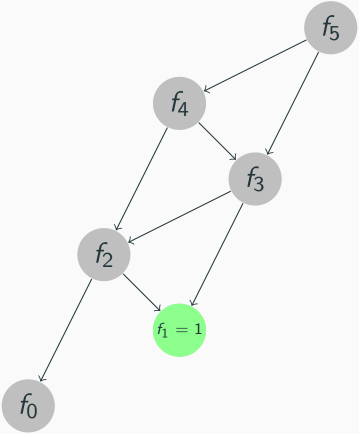

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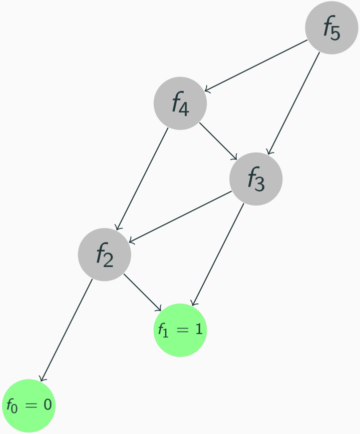

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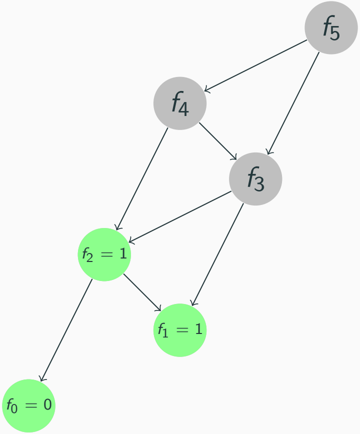

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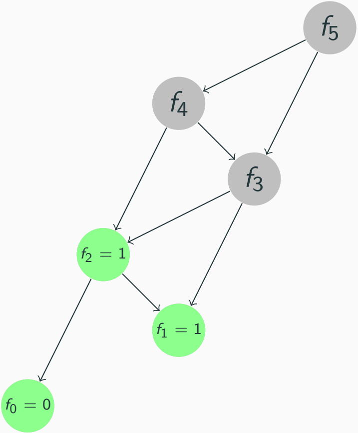

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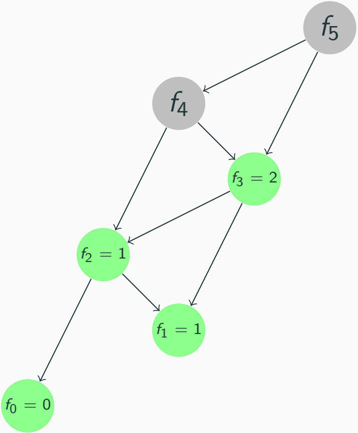

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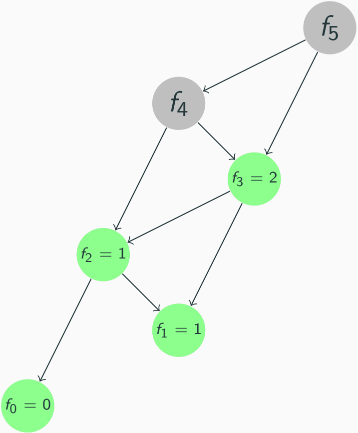

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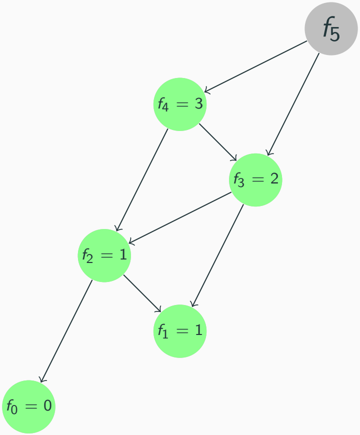
\boldsymbol{f_2} &= \boldsymbol{f_1} + \boldsymbol{f_0} \\
\boldsymbol{f_1} &\boldsymbol{= 1} \\
\boldsymbol{f_0} &\boldsymbol{= 0}
\end{aligned}
$$



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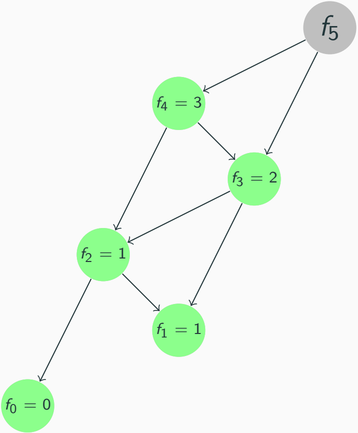

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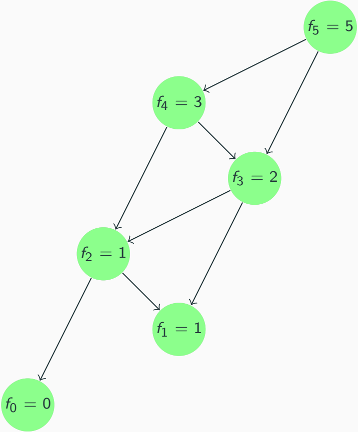

Fibonacci's sequence: The solution

A simple modification of our intuitive algorithm suffices:

```
vector<int> v; // initialized with v.resize(n+1,-1)
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    if(v[n]!=-1) return v[n];
    if(n==0) return v[0] = 0;
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```

Our running time is now down to $\mathcal{O}(n)$.
There's a better solution in $\mathcal{O}(\log(n))$, but it is not in our scope for today.

The recipe for creating a good DP solution

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This was a simple example, but the same schemata apply to much more complicated problems. We shall now generalize what we've learned from Fibonacci's sequence and apply it to other problems.

The four steps of DP

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Here is a classic method of thinking about dynamic programming, using four basic steps.

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Think first, code second!

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Think first, code second!

1. Define subproblems.
2. Find a general recurrence formula to solve a subproblem using the solution to other subproblems.
3. Find the base case(s).
4. Which is the relevant subproblem?

The four steps of DP and Fibonacci's sequence

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How does our solution for Fibonacci's sequence match our four steps?

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1. Suproblems:

The four steps of DP and Fibonacci's sequence

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The four steps of DP and Fibonacci's sequence

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The four steps of DP and Fibonacci's sequence

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3. Base cases: $f_0 = 0, f_1 = 1$.
4. Relevant suproblem: f_n.

How to implement a DP solution

Implementation

— Okay, I’ve followed your four steps. How do I use this to code a solution now?

Implementation

Use a recursive function like you would in an intuitive solution.

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- Call the function for the relevant subproblem.

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Implementation

Use a recursive function like you would in an intuitive solution.

- Call the function for the relevant subproblem.
- At the beginning of each call, check if you have already computed the solution for this subproblem.
- **Store** and return the answer:
    - If you're at a base case, compute the solution for that one.
    - Else, use a recursive call to compute the solution for this subproblem according to the formula.

Another example: Bottles

Bottles: Task statement

The problem is the following:

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- On the second day of the SOI workshop, there are *n* bottles of soda.

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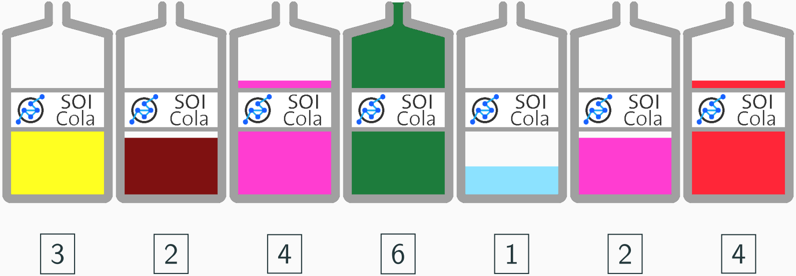
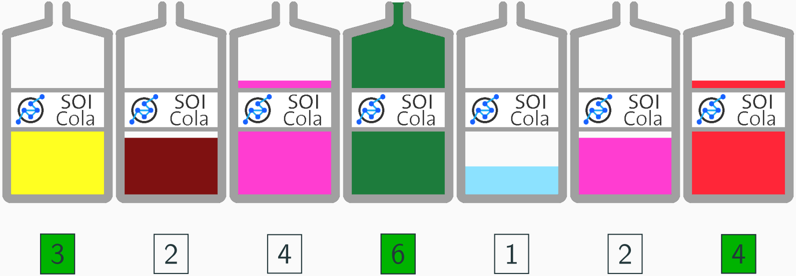
- On the first day, he drank all of the soda and the other participants did not get any.
- The leaders thus ruled that Stofl should be allowed to drink only from non-adjacent bottles.
- Compute the maximum amount of soda that Stofl can drink, given the volumes v_i of all available bottles.

Bottles: Sample case



Bottles: Sample case



Stofl can drink 13 units of soda at most!

Bottles: The four steps of DP

How do we model this problem using the four steps described earlier?

Bottles: Subproblems

Our general subproblem will be

Bottles: Subproblems

Our general subproblem will be s_i, the maximum amount of soda drinkable if the problem is restricted to the $i+1$ first bottles, for any $0 \leq i < n$.

Bottles: General formula

When trying to compute s_i, the solution for the $i + 1$ first bottles, we have two possibilities:

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When trying to compute s_i, the solution for the $i + 1$ first bottles, we have two possibilities:

1. We drink the soda in the i-th bottle (0-based).
2. We do not drink from that bottle.

Bottles: General formula

How does this translate into a formula?

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1. If we do not drink the *i*-th bottle, Stofl can drink

Bottles: General formula

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Bottles: General formula

How does this translate into a formula?

1. If we do not drink the i-th bottle, Stofl can drink s_{i-1}.
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$$s_i = \max(s_{i-1}, s_{i-2} + v_i)$$

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- $s_0 = v_0$ (if there is only one bottle, just drink it).
- $s_1 = max(v_0, v_1)$ (when there are two bottles, you can always drink exactly one of those, so just pick the largest one).

Bottles: Relevant subproblem

The relevant subproblem is the one including all bottles, so the subproblem we want to compute is $s_{\dots}$.

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Bottles: Slow solution

Applying this recusion intuitively gives this code:

```
vector<int> v   ;
int dp(int i) {

    if(i==0) return        v[0];
    if(i==1) return        max(v[0],v[1]);
    return        max(dp(i-1),dp(i-2)+v[i]);
} int main() {
    int n; cin >> n;
    v.resize(n);

    for(int i = 0; i < n; i++) cin >> v[i];
    cout << dp(n-1) << endl;
}
```

Bottles: DP solution

Improving it with memoization is easy:

```
vector<int> v, m;
int dp(int i) {
    if(m[i]!=-1) return m[i];
    if(i==0) return m[0] = v[0];
    if(i==1) return m[1] = max(v[0],v[1]);
    return m[i] = max(dp(i-1),dp(i-2)+v[i]);
} int main() {
    int n; cin >> n;
    v.resize(n);
    m.resize(n,-1);
    for(int i = 0; i < n; i++) cin >> v[i];
    cout << dp(n-1) << endl;
}
```

Bottles: Runtime analysis

This is just like with Fibonacci: we went from an exponential to a linear runtime!

Conclusion

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- For example, it enables you to compute some recursive functions faster, for example Fibonacci's sequence.
- A lot of optimization problems require a dynamic programming solution. A good way to recognize them is that they are about making choices: choosing whether to build a wall or a window, whether to pack an item in one's bag, etc.

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- DP is hard for most people.
- The concept is simple, but applying it to a problem and implementing the solution is difficult.
- Always think before you code!
- Most important of all: solve, solve, solve!

What's next:

Solve DP tasks on the grader.

Next lecture: Subset Sum.